Partially Ordered Sets

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Introduction

A *poset* is a set P equipped with a binary relation denoted $\leq$.

Similar to totally ordered set such as $\mathbb{N}$ or $\mathbb{R}$ except two elements may be *incomparable*.

Posets satisfy the following axioms:

1. Reflexivity: $\forall t \in P, t \leq t$ and $t \geq t$.
2. Anti-Symmetry: If $s \geq t$ and $s \leq t$, then $s = t$.
3. Transitivity: If $s \geq t$ and $t \geq u$, then $s \geq u$.

Cover Relations and Hasse Diagrams

Definition

For $s, t \in P$, we say that s *covers* t if $s > t$ and there is not $u \in P$ such that $s > u > t$.

Definition

A *Hasse Diagram* is an undirected graph visualizing cover relations, with each element as a vertice and an edge between two elements if one covers the other. The greater element is "above" the lesser one.

Examples

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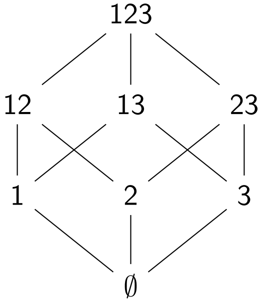
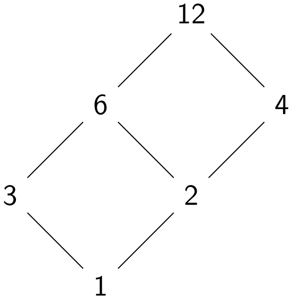


B_3



D_{12}

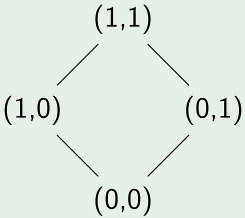
Products

Definition

The product $P \times Q$ of two posets P and Q is the poset of all pairs (s, t) with $s \in P$ and $t \in Q$ with $(s, t) \leq (s', t')$ iff $s \leq s'$ and $t \leq t'$.

Example

The Hasse Diagram for the poset $\mathbf{2} \times \mathbf{2}$ is shown below. It is isomorphic to B_2. In fact, $\mathbf{2}^n \cong B_n$.



Subsets

Definition

A *chain* of length ℓ is a sequence of elements $t_0 < t_1 < t_2 \ldots < t_l$ in P, and a multichain of length ℓ is a sequence of elements $t_0 \leq t_1 \leq t_2 \ldots \leq t_l$.

Definition

An *order ideal* of an element of P is a set of elements I such that if $t \in I$ and $s \leq t$, then $s \in I$.

Definition

An *interval* $[s, t]$ of P is the set of all elements u such that $s \leq u \leq t$. $Int(P)$ is the set of all intervals in P

Incidence Algebras

The Incidence Algebra of a poset P, denoted I_P or just I, is the algebra of all functions $f : Int(P) \to K$, where K is any field.

Multiplication or convolution is defined as:

$$fg(s,t) = \sum_{s \leq u \leq t} f(s,u)g(u,t)$$

The identity δ (or sometimes 1) is:

$$\delta(s,t) = \begin{cases} 1 & s = t \\ 0 & s \neq t \end{cases}$$

The Zeta Function

The *zeta* function ζ is defined by $\zeta(s,t) = 1$ for all $s \leq t \in P$.

We can see that

$$\zeta^2(s,t) = \sum_{s \leq u \leq t} \zeta(s,u)\zeta(u,t) = \sum_{s \leq u \leq t} 1 = \#[s,t]$$

or more generally

$$\zeta^n(s,t) = \sum_{s=t_0 \leq t_1 \ldots \leq t_n = t} 1 = \#\text{ of multichains of length } n \text{ from } s \text{ to } t$$

We can also show

$$(2-\zeta)^{-1}(s,t) = \text{total number of chains from } s \text{ to } t$$

The Möbius Function

Definition

The Möbius function μ (or μ_P for a specific poset) is the inverse of ζ, so $\mu\zeta(s,t) = \zeta\mu(s,t) = \delta(s,t)$. It is defined explicitly as

$$\mu(s,t) = \begin{cases} 1 & s = t \\ -\sum_{s \leq u < t} \mu(s,u) & s < t \end{cases}$$

Theorem

Let P and Q be finite posets and let $P \times Q$ be their product poset. Let $s, s' \in P$ and $t, t' \in Q$ such that $(s,t) \leq (s',t')$ in $P \times Q$. Then

$$\mu_{P\times Q}((s,t),(s',t')) = \mu_P(s,s')\mu_Q(t,t')$$

Möbius Inversion

Theorem

If P is a poset such that every order ideal is finite, and f, g are two functions from $P \to K$, where K is any field, then

$$f(t) = \sum_{s \leq t} g(s)$$

if and only if:

$$g(t) = \sum_{s \leq t} f(s)\mu(s,t)$$

for all $t \in P$.

We also have the dual version:

$$f(t) = \sum_{s \geq t} g(s) \text{ if and only if: } g(t) = \sum_{s \geq t} f(s)\mu(s,t)$$

We can quickly calculate that $\mu_{\mathbf{2}}(0,1) = -1$. Since $\mathbf{2}^n \cong B_n$, we calculate that

$$\mu_{B_n}(s,t) = \prod_i \mu_{\mathbf{2}}(s_i, t_i) = (-1)^{\#(s_i \neq t_i)}$$

where $s_i = 1$ if $i \in s$ and 0 if not (t_i is defined analogously)

As sets this means $\mu_{B_n}(S,T) = (-1)^{\#(S-T)}$. Dual Möbius inversion then gives us:

$$f(T) = \sum_{S \supseteq T} g(S) \text{ if and only if: } g(T) = \sum_{S \supseteq T} f(S)(-1)^{\#(S-T)}$$

This statement is equivalent to the Principle of Inclusion-Exclusion.

Number Theory

If $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$, then

$$D_n \cong (a_1 + 1) \times (a_2 + 1) \times \dots \times (a_k + 1)$$

Easy computation gives us

$$\mu(s, n) = \mu\left(\frac{n}{s}\right) = \begin{cases} (-1)^p & \text{if } \frac{n}{s} \text{ is square-free} \\ 0 & \text{otherwise} \end{cases}$$

Where p is the number of prime factors of $\frac{n}{s}$. Möbius inversion gives

$$f(n) = \sum_{s|n} g(s) \text{ if and only if: } g(n) = \sum_{s|n} f(s)\mu\left(\frac{n}{s}\right)$$

This formula is known as the Möbius transform.